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Diagonal Operators, Approximation Numbers, and Kolmogoroff Diameters

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INTRODUCTION

This paper is a study of the approximation numbers and Kolmogoroff diameters of diagonal mappings (primarily on the l_p-spaces).

In Section 1, we make some remarks concerning the relationship between the approximation numbers and Kolmogoroff diameters of arbitrary operators. Of special interest here is Theorem 1.2 and subsequent remarks that roughly state that the approximation numbers and Kolmogoroff diameters are surjectives of one another in the sense of Grothendieck [9, 10]. The most important result in Section 1 is Theorem 1.8, which gives the exact values of

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the approximation numbers and Kolmogoroff diameters for diagonal mappings from l_∞ to l_p, $1 \leqslant p < \infty$. Using 1.8, we show how to construct operators whose Kolmogoroff diameters and approximation numbers satisfy various growth conditions.

In Section 2, we compute exactly or give asymptotic estimates for the approximation numbers and Kolmogoroff diameters of diagonal mappings from l_p to l_q, $1 \leqslant p, q \leqslant \infty$, essentially completing a study initiated by Pietsch [36, 37]. Section 2 also provides the calculations necessary for later sections.

In Section 3, we consider some special kinds of Schauder bases in Banach spaces. The principal result of this section is Theorem 3.4, which, in a certain sense, is a generalization of a result of Marcus [26] concerning the so-called H-operators. Using a previous construction of Morrell and Retherford [31] we are able to show (Theorem 3.5) that there are always nontrivial subspaces of infinite-dimensional Banach spaces that satisfy the hypotheses of 3.4.

In the final section, motivated by a classical result of Bernstein [1], we introduce the concept of a Bernstein pair and prove that any pairing of "classical" Banach spaces forms a Bernstein Pair.

Standard Concepts

All spaces considered are Banach spaces. By *operator* or *mapping* we mean a bounded linear transformation. We denote by $\mathscr{L}(E, F)$ the operators from E to F and by $K(E, F)$ the compact operators from E to F. Also by $\mathscr{F}_0(E, F)$ we denote the finite-rank operators from E to F and by $\mathscr{F}(E, F)$ the closure of $\mathscr{F}_0(E, F)$ in $\mathscr{L}(E, F)$. By an *isomorphism*, we mean an open, one-to-one mapping. A *projection* P is a member of $\mathscr{L}(E, E)$ such that $P^2 = P$. If A is a subspace (i.e., closed linear subspace) of E, then A is *complemented* in E if there is a projection $P \in \mathscr{L}(E, E)$ with $P(E) = A$.

If $\{x_\alpha\} \subset E$, then by $[x_\alpha]$ we denote the closed linear span of $\{x_\alpha\}$ in E.

By a *biorthogonal system* (x_i, f_i) in E, we mean sequences $(x_i) \subset E$, $(f_i) \subset E^*$ such that

$$f_i(x_j) = \delta_{ij}.$$

A *Schauder basis* for E is a biorthogonal system (x_i, f_i) such that for each $x \in E$

$$\sum_{i=1}^{\infty} f_i(x)\, x_i = x,$$

convergence in the norm of E. A sequence $(x_n) \subset E$ is a *basic sequence* if it is a basis for $[x_n]$. A basis (x_n, f_n) is *shrinking* if (f_n) is a basis for E'.

For $1 \leqslant p \leqslant \infty$, we denote by l_p the Banach space of scalar sequences $a = (a_i)$ with

$$\| a \| = \left(\sum_{i=1}^{\infty} | a_i | p\right)^{1/p}, \qquad \text{if} \quad 1 \leqslant p < +\infty,$$
$$= \sup_i | a_i |, \qquad \text{if} \quad p = \infty.$$

Also, by c_0, we denote the closed subspace of l_∞ consisting of all null sequences.

Let $(x_i) \subset E$, $1 \leqslant p \leqslant +\infty$, and let p' be given by $(1/p) + (1/p') = 1$. (Of course, for $p = \infty$, we understand $p' = 1$ and for $p = 1$, $p' = \infty$.) We say that (x_i) is *ϵ_p-finite* provided

$$\epsilon_p(x_i) = \sup \left\{ \left(\sum_{i=1} | f(x_i)|^p\right)^{1/p} : f \in E^*, \|f\| \leqslant 1 \right\}$$

is finite.

Approximation Numbers and Kolmogoroff Diameters

Let $A_n(E, F)$ denote the operators of rank at most n from E to F. Following Pietsch [36, 37], we define the nth *approximation number*, $\alpha_n(T)$, $T \in \mathscr{L}(E, F)$, as follows:

$$\alpha_n(T) = \inf\{\| T - A \|: A \in A_n(E, F)\}.$$

For a Banach space X, we let U_X denote the unit ball of X. Now, we recall the definition of the nth Kolmogoroff diameter, $d_n(A)$, of a bounded set $A \subset X$ with respect to U_X (see [17, 18, 30])

$$d_n(A) = d_n(A, U_X) = \inf_L[\inf\{\epsilon > 0 : L + \epsilon U_X \supset A\}],$$

where the infimum is taken over all at most n-dimensional subspaces L of F.

For $T \in \mathscr{L}(E, F)$ we define the nth Kolmogoroff diameter of T, $d_n(T)$, by

$$d_n(T) = d_n(TU_E, U_F).$$

It is easy to see that $d_n(T) = \inf \| qT \|$, where the infimum is over all quotient maps $q: F \to F/F_0$, where $\dim F_0 \leqslant n$. Clearly, $\alpha_n(T)$ is the value of best linear approximation and $d_n(T)$ is the value of best nonlinear approximation to T (in the sense of finite-rank operators). Also, it is clear that $\alpha_n(T)$ and $d_n(T)$ are monotone decreasing sequences and that $\lim_n \alpha_n(T) = 0$ if and only if $T \in \mathscr{F}(E, F)$ and $\lim_n d_n(T) = 0$ if and only if $T \in K(E, F)$.

For a brief discussion of the algebraic and analytic properties of $\alpha_n(T)$ and $d_n(T)$, see [30]. For operators on Hilbert spaces the behavior of $\alpha_n(T) =$

$d_n(T)$ has been extensively studied. Most of the results concerning these characteristics are compiled in the book of Gohberg and Krein [8]. For arbitrary Banach spaces there are few results. (See the remarks at the end of the paper.) Two of the best references are the papers of Marcus and Macaev [26, 27]; related results are also to be found in the classic Memoir of Grothendieck, Part 2].

The Krein–Milman–Krasnoselskii Theorem

The following property, established by Krein, Milman, and Krasnoselskii [20] is very useful in computing Kolmogoroff diameters:

Let M be an $(n+1)$-dimensional subspace of a Banach space E. Then, $d_k(U_E \cap M, U_E) = 1$ for all $k \leqslant n$.

We give a short proof of the above property. Our proof is taken from the monograph of Bessaga and Retherford [2]. A similar proof is given in [25]. For another proof, see Tihomirov [43].

First, we make the following observation. If $\langle X, \|\ \|\rangle$ is a normed linear space and M a finite-dimensional subspace of X, then for each $\epsilon > 0$ there is a norm $|\ |$ such that for each $x \in X$

$$\| x \| \leqslant | x | \leqslant (1+\epsilon) \| x \|$$

and with respect to $|\ |$, the metric projection onto M is unique.

Indeed if $\dim M = n$, choose $f_1 \cdots f_n$, $\| f_i \| = 1$, biorthogonal to some basis for M and let

$$| x | = \| x \| + \left[n^{-1}\epsilon \left(\sum_{i=1}^{n} | f_i(x)|^2 \right)^{1/2} \right].$$

The proof of the Krein, Milman, Krasnoselskii theorem now proceeds as follows: The unit sphere S_M, of M, is homeomorphic to the Euclidean sphere S^n. If L is a k-dimensional subspace of E, introduce the norm described above and let $f(x)$ be the best approximation to x from L. Then, on S_M, f is a well-defined continuous antipodal map and so by Borsuk's antipodal mapping theorem [45] there is an $x_0 \in S_M$ such that $f(x_0) = 0$, i.e., $d_k(M \cap U_E, U_E) = 1$.

We will make use of the following form of the Krein, Milman, Krasnoselskii theorem:

Let M_n and M_{n+1} be n-dimensional and $(n+1)$-dimensional subspaces of a normed linear space E. Then, there is a $y_0 \in M_{n+1}$ such that

$$d(y_0, M_n) = \inf\{\| y_0 - z \| : z \in M_n\} = \| y_0 \|.$$

p-absolutely summing operators, nuclear operators, and maps of type l_p:

For a sequence (x_n) in a Banach space E we set

$$\gamma_p(x_n) = \left(\sum \| x_n \|^p\right)^{1/p}, \qquad \text{if} \quad 1 \leqslant p < +\infty$$
$$= \sup \| x_n \|, \qquad \text{if} \quad p = \infty.$$

An operator $T \in \mathscr{L}(E, F)$ is *p-absolutely summing* [33–35] if there is a constant C such that

$$\gamma_p(Tx_i) \leqslant C\epsilon_p(x_i)$$

for all finite sets $\{x_i\}_{i=1}^N$ in E.

A $T \in \mathscr{L}(E, F)$ is *nuclear* [9] provided that there are $(f_n) \subset E'$, $(y_n) \subset F$ such that

$$T = \sum_{n=1}^{\infty} f_n \otimes y_n \qquad \text{and} \qquad \sum_{n=1}^{\infty} \| f_n \| \| y_n \| < +\infty.$$

Here, $f_n \otimes y_n$ denotes the rank one operator $f_n \otimes y_n(x) = f_n(x)\, y_n$.

An operator $T \in \mathscr{L}(E, F)$ is of *type* l_p [36, 37], if $(\alpha_n(T)) \in l_p$, where $(\alpha_n(T))$ is the sequence of approximation numbers of T. These operators generalize the classes σ_p of von Neumann–Schatten [46].

Diagonal operators

Let E and F be Banach spaces with Schauder bases (x_n) and (y_n), respectively. An operator $T \in \mathscr{L}(E, F)$ with $Tx_n = \lambda_n y_n$, where (λ_n) is some fixed scalar sequence, is called a diagonal operator (with respect to (x_n) and (y_n)). In this work, we will be primarily concerned with diagonal mappings from l_p to l_q with respect to the usual unit vector bases of these spaces. Also, we will assume that $| \lambda_1 | \geqslant | \lambda_2 | \geqslant \cdots$. In our computations there will be no loss of generality in making this assumption. We will make this precise later. "$T \sim (\lambda_n)$" will mean "a diagonal map T corresponding to (λ_n)." We will also call a mapping on l_∞ defined by coordinate-wise multiplication a diagonal map.

The $\mathscr{L}_p$-spaces

In the final section of this paper, we will consider the $\mathscr{L}_p$-spaces of [23, 24]. If E and F are isomorphic Banach spaces, the *distance coefficient* of E and F, $d(E, F$, is given by

$$d(E, F) = \inf \| T \| \| T^{-1} \|,$$

where the infimum is over all isomorphisms from E onto F.

Let $\lambda \geqslant 1$ and $1 \leqslant p \leqslant \infty$. A Banach space E is an $\mathscr{L}_{p,\lambda}$-space if for each finite-dimensional subspace $F \subset E$, there is a finite-dimensional subspace B with $F \subset B \subset E$ such that

$$d(B, l_p^n) \leqslant \lambda, \qquad n = \dim B,$$

the dimension of B. Here, l_p^n is the space of n-tuples with the l_p-norm. Finally, a space is an $\mathscr{L}_p$-space if it is an $\mathscr{L}_{p,\lambda}$-space for some $\lambda \geqslant 1$ [23]. These spaces include and generalize the classical $L_p(S, \Sigma, \mu)$ spaces and the $C(K)$-spaces.

We use the following result of [23]:

For $1 \leqslant p < \infty$ an $\mathscr{L}_p$-space contains a complemented subspace isomorphic to l_p.

Local Reflexivity and the Approximation Property

We will also need the following result due to Lindenstrauss and Rosenthal [24; see also 39] (the principle of local reflexivity): *Let X be a Banach space (regarded as a subspace of X''), let U and F be finite-dimensional subspaces of X'' and X', respectively, and let $\epsilon > 0$. Then, there is a one-to-one operator $T: U \to X$ with $Tx = x$ for $x \in X \cap U$, $f(Te) = e(f)$ for $e \in U$ and $f \in F$ and $\| T \| \|T^{-1}\| < 1 + \epsilon$.*

Finally, a Banach space E has the *approximation property* if $\mathscr{F}(F, E) = K(F, E)$ for every Banach space F.

It is known [5, 6], that not every Banach space has the approximation property. For numerous equivalent formulations of the approximation property, see [9].

1. Relationships Between $d_n(T)$ and $\alpha_n(T)$

Kadec and Snober [16], using a result of John [14], have shown that for any n-dimensional subspace E_n of a Banach space E, there is a projection from E onto E_n of norm at most $n^{1/2}$. Using this result, it is easy to strengthen a result of Pietsch [36].

1.1. Theorem. *For any $T \in \mathscr{L}(E, F)$ the following inequality is valid*

$$d_n(T) \leqslant \alpha_n(T) \leqslant (n^{1/2} + 1)\, d_n(T), \qquad \textit{for each } n.$$

Proof. Let $A \in A_n(E, F)$ and $\delta = \| T - A \|$. If $\| x \| \leqslant 1$, $Tx \in \delta U_F + Ax$ and so $T(U_E) \subset \delta U_F + A(E)$, i.e., $d_n(T) \leqslant \alpha_n(T)$.

Now, suppose that $T(U_E) \subset \beta U_F + L_n$, $\dim L_m \leqslant n$. Let $P_n : F \to L_n$

be a projection with $\| P_n \| \leqslant n^{1/2}$ and let $A_n = P_n T$. Then, since $Tx = y + z$, where $y \in L_n$ and $\| z \| \leqslant \beta$ for $x \in U_E$, we have

$$\| Tx - A_n x \| = \|(y + z) - P_n(y + z)\| = \| z - P_n(z)\| \\ \leqslant (1 + n^{1/2})\, \beta.$$

Thus, $\alpha_n(T) \leqslant \| T - A_n \| \leqslant (1 + n^{1/2})\, \beta$ and since β was arbitrarily chosen with the above property

$$\alpha_n(T) \leqslant (1 + n^{1/2})\, d_n(T).$$

Some remarks concerning Theorem 1.1 are in order. The best value, $p(n)$, for which

$$\alpha_n(T) \leqslant p(n)\, d_n(T)$$

is not known. However, in general, $p(n)$ cannot be replaced by a constant K, independent of n. Indeed, Enflo [5], (see also Figiel [6]) has constructed a Banach space that lacks the approximation property. Thus, for such a space E, there exists a Banach space F and a $T \in \mathscr{L}(F, E)$ such that $\lim_{n\to\infty} d_n(T) = 0$ and $\lim_n \alpha_n(T) \neq 0$.

If $T \in \mathscr{L}(E, F)$, E and F arbitrary Banach spaces, it is easy to see that

$$\lim_{n\to\infty} d_n(T) \leqslant d(T, K(E, F)) = \inf\{\| T - S \| : S \in K(E, F)\} \leqslant d(T, \mathscr{F}(E, F)) \\ = \inf\{\| T - A \| : A \in \mathscr{F}(E, F)\} = \lim_{n\to\infty} \alpha_n(T).$$

Thus, $\lim_n \alpha_n(T) = \lim_n d_n(T)$ for all $T \in K(F, E)$ and all F if and only if E has the approximation property.

Even though d_n and α_n can behave as above, they are, of course, closely related. Our next two results spell out this relationship.

1.2. Theorem. *For any $T \in \mathscr{L}(l_1, E)$, $\alpha_n(T) = d_n(T)$ for each n.*

Proof. Suppose that $T(U_{l_1}) \subset \delta(U_E) + F_n$, $\dim F_n \leqslant n$. Let (e_n) denote the unit vector basis of l_1. Then, $Te_m = \delta v_m + x_m$, where $\| v_m \| \leqslant 1$ and $x_m \in F_n$. Define $A: l_1 \to E$ by $Ae_m = x_m$. Then, A is well-defined since (x_m) is bounded. Thus,

$$\| T - A \| = \sup_m \| Te_m - Ae_m \| \leqslant \delta,$$

and since rank $A \leqslant n$ we have $\alpha_n(T) \leqslant d_n(T)$.

Obviously, Theorem 1.2 extends to $l_1(\Gamma)$ domains for any set Γ. Less obvious is the fact that 1.2 also extends to $L_1(\mu)$ and to *separable* $\mathscr{L}_1$-spaces. This follows from the remarks in [39] and the fact that $L_1(\mu)$ spaces have the lifting property. We conjecture that for a $\mathscr{L}_{1,\lambda}$-domain, we always obtain $\alpha_n(T) \leqslant C(\lambda)\, d_n(T)$.

We remark that the conjecture is true for compact T. Indeed, if E is an $\mathscr{L}_{1,\lambda}$-space and $S: E \to F/F_0$ is compact, then there is a number $C(\lambda)$ and an operator $\tilde{S}: E \to F$ such that $\tilde{S}$ is compact, $Q_0\tilde{S} = S$ and $\| \tilde{S} \| \leqslant C(\lambda) \| S \|$ (see [24]). Thus, if $T: E \to F$ is compact, $\epsilon > 0$ and $n \geqslant 1$, choose $F_0 \subset F$, $\dim F_0 \leqslant n$ such that $\| Q_0 T \| \leqslant d_n(T) + \epsilon$. Then, $Q_0 T$ is compact and so has a compact lifting $\tilde{T}$ with $\| \tilde{T} \| \leqslant C(\lambda) \| Q_0 T \|$. If $A = T - \tilde{T}$, then $Q_0(\tilde{T} - T) \equiv 0$ and so $\tilde{T} - T(x) \in F_0$, i.e., Rank $\tilde{T} - T \leqslant n$. Thus,

$$\alpha_n(T) \leqslant \| T - (T - \tilde{T}) \| = \| \tilde{T} \| \leqslant C(\lambda) \| Q_0 T \| \leqslant C(\lambda)[d_n(T) + \epsilon].$$

Our next result, an immediate corollary to Theorem 1.2, shows the intimate relationship between d_n and α_n. It also points out the difficulty in computing $p(n)$ above.

1.3. COROLLARY. *Let E be a Banach space and q a metric surjection of $L_1(\mu)$ onto E. Then, for any $T \in \mathscr{L}(E, F)$, $d_n(T) = \alpha_n(Tq)$ for each n.*

Proof. From the definition, it is clear that $d_n(T) = d_n(Tq)$.

Before proceeding to the next result, we recall that Pietsch has shown [36] that every operator of type l_1 is nuclear.

1.4. COROLLARY. *For any $\epsilon > 0$, if $\lim_{n\to\infty} n^{3/2+\epsilon} d_n(T) = 0$, then T is of type l_1 (hence nuclear).*

The result 1.4 is immediate from 1.1. Corollary 1.4 answers a question of Mitiagin [28] and strengthens some results in [2, 38]. We now give an example showing that there is no converse to Corollary 1.4.

1.5. EXAMPLE. Let $\lambda_n = 1/n^{1/2}[\log n]$ and let D be the diagonal operator from l_1 to l_2 given by $De_n = \lambda_n e_n$, where (e_n) denotes the unit vector basis in both l_1 and l_2. To see that D is nuclear, observe that we have the following factorization:

$$\begin{tikzcd} l_1 \arrow[rr, "D"] \arrow[dr, "i"'] & & l_2 \\ & l_2 \arrow[ur, "D_1"'] & \end{tikzcd}$$

where i is the natural inclusion map and D_1 is the diagonal mapping on l_2 corresponding to (λ_n). Since $(\lambda_n) \in l_2$, D_1 is a Hilbert–Schmidt operator [36] and i is 1-absolutely summing [9]. Thus, by [9] (or [36]), D is nuclear.

In the next section, we prove a result which computes $d_n(T)$ exactly. However, for our present purposes, it suffices to show that $\alpha_n(D) \geqslant (\lambda_{n+1}/(n+1)^{1/2}$, since by 1.3, $d_n(D) = \alpha_n(D)$ and we then have $(n+1)^{3/2+\epsilon} d_n(D) \geqslant (n+1)^{\epsilon}/\log(n+1)$. To see that $\alpha_n(D) \geqslant \lambda_n/(n+1)^{1/2}$, observe that if A is any operator from l_1 to l_2 with rank $A \leqslant n$, then there is a vector $\xi = (\xi_i)$ with $\xi_i = 0$ for $i > n+1$, $\sum_{i=1}^{n+1} |\xi_i| = 1$ such that $A(\xi) = 0$. Thus,

$$\|D - A\| \geqslant \|(D-A)(\xi)\| = \left(\sum_{i=1}^{n+1} |\lambda_i|^2 |\xi_i|^2\right)^{1/2} \geqslant |\lambda_{n+1}| \left(\sum_{i=1}^{n+1} |\xi_i|^2\right)^{1/2}$$
$$\geqslant \lambda_{n+1}/(n+1)^{1/2}.$$

We now present some results that show considerably more than Example 1.5. In the next section, we compute (or give asymptotic estimates of) the approximation numbers of diagonal operators on arbitrary l_p-spaces. But, since the techniques used to obtain the approximation numbers of diagonals from l_∞ to l_p is so radically different from those employed in Section 3, we give that result here.

We need two lemmas. The proof of the first lemma is immediate.

1.6. Lemma. *If P is an n-dimensional polytope in Euclidean n-space, then its boundary is the union of its $(n-1)$-dimensional faces.*

1.7. Lemma. *Let P be an n-dimensional polytope in Euclidean n-space and let V be a k-dimensional manifold, $1 \leqslant k \leqslant n$. If $P \cap V \neq \varnothing$, then there is an $(n-k)$-dimensional face F of P such that $F \cap V \neq \varnothing$.*

Proof. The proof is by induction. The result is obvious for $n = 1$, so suppose the lemma holds for $n-1$. By Lemma 1.6, $P \cap V \neq \varnothing$ implies that there is an $(n-1)$-dimensional face F_1 of P such that $V \cap F_1 \neq \varnothing$. If U_1 denotes the affine manifold spanned by F_1, then $V \cap V_1$ is an affine manifold of V_1 and the dimension of $V \cap V_1 \geqslant k-1$. Without loss of generality, we can suppose that $k > 1$. Let U_2 be an affine manifold of $V \cap V_1$ of dimension $k-1$ that intersects F_1. Now, F_1 is an $(n-1)$-dimensional polytope and so by the induction hypothesis, there is an $(n-1)-(k-1) = (n-k)$-dimensional face F of F_1 such that $F \cap V_2 \neq \varnothing$. But F is also an $(n-k)$-dimensional face of P and $F \cap V \neq \varnothing$.

Now we can prove the main result of this section. This result improves the result [37] of Pietsch.

1.8. Theorem. *Let $1 \leqslant p < \infty$ and let $T\colon l_\infty \to l_p$ or $c_0 \to l_p$ be a diagonal map $(a_n) \to (\lambda_n a^n)$, where $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$. Then, $\alpha_k(T) = d_k(T) = (\sum_{i=k+1}^{\infty} |\lambda_i|^p)^{1/p}$ for each k.*

Proof. It is clear that for T to be well defined, $(\lambda_n) \in l_p$. Also, $\alpha_k(T) \leqslant (\sum_{k+1}^{\infty} |\lambda_i|^p)^{1/p}$ for any k by considering the kth truncation of T.

To prove the opposite inequality, let $A: l_\infty \to l_p$ be an arbitrary operator of rank $\leqslant k$, say $A = \sum_{j=1}^{k} f_i \otimes y_i$. Then,

$$\| T - A \| = \sup_{\|x\| \leqslant 1} \left(\sum_{n=1}^{\infty} | \lambda_n x_n - \sum_{j=1}^{k} \langle x, f_j \rangle y_{jn} |^p \right)^{1/p},$$

where $y_j = (y_{jn}) \in l_p$ and $x = (x_n) \in l_\infty$. Let $m > k$ and let

$$V = \{x \in l_\infty{}^m : \langle x, f_j \rangle = \text{ for } 1 \leqslant j \leqslant k\}.$$

Then, V is a subspace of $l_\infty{}^m$ of dimension $\geqslant m - k$ that intersects the unit ball of $l_\infty{}^m$. By Lemma 1.7, V intersects a k-dimensional face of the unit ball of $l_\infty{}^m$. Thus, there exists $x \in l_\infty{}^m$ with $\| x \| = 1$ and indices $i_1, ..., i_{m-k}$, $1 \leqslant i_j \leqslant m$ for each j, such that $| x_{i_j} | = 1$ for $1 \leqslant j \leqslant m - k$. Consider $\hat{x} = (x_i) \in l_\infty$, where $x_i = 0$ for $i \geqslant m + 1$ and agrees with $x \in l_\infty{}^m$ above, for $i \leqslant m$. Then

$$\left(\sum_{n=1}^{\infty} | \lambda_n x_n - \sum_{j=1}^{k} \langle x, f_j \rangle y_{jn} |^p \right)^{1/p} = \left(\sum_{n=1}^{m} | \lambda_n x_n |^p \right)^{1/p} \geqslant \left(\sum_{j=1}^{m-k} | \lambda_{n_j} |^p \right)^{1/p}$$

$$\geqslant \left(\sum_{i=k+1}^{m} | \lambda_i |^p \right)^{1/p}.$$

Since A and m were arbitrary we conclude that

$$\alpha_k(T) \geqslant \left(\sum_{k+1}^{\infty} | \lambda_i |^p \right)^{1/p}.$$

To see that $d_k(T) \geqslant (\sum_{k+1}^{\infty} | \lambda_i |^p)^{1/p}$ and hence, equal to $\alpha_k(T)$, one only has to apply the Krein–Milman–Krasnoselskii theorem and proceed as above.

From 1.8 we obtain as a corollary the following unpublished result of Macaev (attributed by Marcus [26]).

1.9. Corollary. *Let (β_n) be an l_1 sequence with $\beta_n \geqslant \beta_{n+1} \geqslant 0$. Let K denote the set of all elements $(a_n) \in l_1$ such that $| a_n | \leqslant b_n$ for each n. Then, $d_n(K) = (\sum_{n+1}^{\infty} \beta_n)$.*

Proof. Observe that $K = T(U_{l_\infty})$, where T is the diagonal determined by (β_n).

Using 1.8, it is now easy to construct a nuclear operator T with $\alpha_n(T) = d_n(T)$ tending to 0 as slowly as we please.

1.10. EXAMPLE. Let (λ_n) be a positive sequence monotonically tending to 0 and let $\beta_n = \lambda_n - \lambda_{n+1}$. Define $T: l_\infty \to l_1$ by $T(x) = (\beta_n x_n)$. Then, T is nuclear and $d_n(T) = \alpha_n(T) = \lambda_{n+1}$ for each n.

Proof. The operator T is clearly a lattice bounded (in the natural order of l_1) operator and hence, [9] nuclear. By 1.8,

$$d_n(T) = \alpha_n(T) = \sum_{n+1}^{\infty} \beta_i = \lambda_{n+1}.$$

The existence of such an operator T is mentioned without proof by Marcus [26].

2. DIAGONAL OPERATORS ON THE l_p-SPACES

In this section, we compute the approximation numbers of diagonal mapping between the l_p-spaces. Before proceeding to our results, we make some comments concerning Theorem 1.8 that also pertain to all of the results in this section.

Suppose that $T: l_\infty \to l_p$ is a diagonal mapping corresponding to a sequence (λ_n) where (λ_n) is *not* necessarily monotone. Let $\Sigma(n)$ denote the collection of all subsets of the positive integers consisting of exactly n elements. Then,

$$\alpha_n(T) = \inf \left\{ \left\{ \left(\sum_i | \lambda_i |^p \right)^{1/p} : i \notin \sigma \right\} : \sigma \in \sum (n) \right\}. \tag{*}$$

For this reason, we can assume that always, (λn) is monotone decreasing in all succeeding calculations. (The modification of (*) in all cases will be apparent.) The proof of (*) follows from the proof of 1.8 and the calculation in [36].

We begin this section by computing the approximation numbers of the natural injections $I: l_p \to l_q$, where $1 \leqslant p \leqslant q \leqslant \infty$.

2.1. THEOREM. *For $I: l_1 \to l_\infty$, $d(I) = \frac{1}{2}$ for every n.*

Proof. To show that $\alpha_n(T) \leqslant \frac{1}{2}$ for all n, it suffices to show that $\alpha_1(I) \leqslant \frac{1}{2}$. Let $e_0 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2},...) \in l_\infty$ and let $A: l_1 \to l_\infty$ be the rank one operator $A(x_i) = (\sum_{i=1}^{\infty} x_i)\, e_0$ for $(x_i) \in l_1$. Then, $\| I - A \| = \sup_i \|(I - A)\, e_i \|_\infty = \sup_i \| e_i - e_0 \|_\infty = \frac{1}{2}$. Thus, $\alpha_1(I) \leqslant \frac{1}{2}$.

Now, suppose that $\alpha_k(I) < \frac{1}{2}$ for some k. Let $\epsilon > 0$ be such that $\alpha_k(I) < \frac{1}{2} - \epsilon$ and choose an operator $A: l_1 \to l_\infty$ of rank at most k such that $\| I - A \| \leqslant \alpha_k(I) + \epsilon/2 < \frac{1}{2} - \epsilon/2$. Then

$$\sup_i \|(I - A)\, e_i \|_\infty < \tfrac{1}{2} - \epsilon/2.$$

If $Ae_i = (a_{ij})_{j=1}^{\infty} \in l_\infty$, then $Ae_i \in B(A)$, where

$$B(A) = \{y \in A(l_1): \| y \|_\infty \leqslant \| A \|\}.$$

Since A has finite rank, $B(A)$ is relatively compact. Now, for each i we have

$$| 1 - a_{ii} | < 1/2 - \epsilon/2 \qquad \text{and} \qquad | a_{ij} | < 1/2 - \epsilon/2 \qquad \text{for } j \neq i.$$

Thus, if $i \neq n$,

$$\begin{aligned} \| Ae_i - Ae_n \|_\infty &= \sup_j | a_{ij} - a_{nj} | \geqslant | a_{ii} - a_{ni} | \\ &\geqslant ([1/2 + \epsilon/2] - [1/2 - \epsilon/2]) = \epsilon. \end{aligned}$$

Thus, $(Ae_i) \subset B(A)$ has no convergent subsequence. Thus, $\alpha_n(I) \geqslant 1/2$ for each n.

On the other hand, it is obvious that $\alpha_n(J) = d_n(J) = 1$ for J the natural injection $l_1 \to c_0$. Since $J^* = I$, we see that, in general, $\alpha_n(T) \neq \alpha_n(T^*)$.

In view of 2.1 the next result is somewhat surprising.

2.2. Theorem. *If* $1 < p \leqslant \infty$ *and* $I: l_p \to l_\infty$ *is the natural injection operator, then* $\alpha_k(I) = 1$ *for all* k.

Proof. Clearly, $\alpha_k(I) \leqslant 1$ for all k. If $\alpha_k(I) < 1$ for some k, choose $\epsilon > 0$ and an operator $A: l_p \to l_\infty$ of rank at most k such that $\| I - A \| < 1 - \epsilon/2$. We can represent A in the following fashion:

$$A = \sum_{i=1}^{k} f_i \otimes y_i, \qquad \text{where} \quad f_i \in l_{p'} \ (1/p + 1/p' = 1) \qquad \text{and} \qquad (y_i) \subset l_\infty .$$

By the choice of A, we have

$$\sup_j \|(I - A)\, e_j \|_\infty \leqslant \| I - A \| < 1 - \epsilon/2.$$

If $y_i = (y_{ij})$ then

$$\left| 1 - \sum_{i=1}^{k} \langle e_j , f_i \rangle\, y_{ij} \right| < 1 - \epsilon/2 \qquad \text{for all } j.$$

Thus,

$$\sum_{i=1}^{k} |\langle e_j , f_i \rangle| < \epsilon/2M, \qquad \text{for all } j,$$

where $M = \max_{1 \leqslant i \leqslant k} \| y_i \|_\infty$. Since $f_i \in l_p$, there exists an index j_0 such that

$$\left(\sum_{j=j_0}^{\infty} |\langle e_j , f_i \rangle|^{p'} \right)^{1/p'} < \epsilon/2kM, \qquad \text{for} \quad 1 \leqslant i \leqslant k.$$

In particular, $|\langle e_{j_0}, f_i\rangle| < \epsilon/2kM$, for all i, $1 \leqslant i \leqslant k$. Thus,

$$\epsilon/2M < \sum_{i=1}^{k} |\langle e_{j_0}, f_i\rangle| < k \cdot (\epsilon/2kM) = \epsilon/2M.$$

This contradiction shows that $\alpha_k(I) \geqslant 1$ for all k.

Our next result shows that, in general, $\alpha_n(T) \neq d_n(T)$.

2.3. THEOREM. *Let $1 < p < \infty$ and $I: l_p \to l_\infty$ the natural injection. Then, $d_n(I) = 2^{-1/p}$ for all n.*

Proof. Let $1 < p < \infty$ and $I: l_p \to l_\infty$. We use the formulation $d_n(I) = \inf\{\| Q_F I \|: F$ is an n-dimensional subspace of l_∞ and $Q_F : l_\infty \to l_\infty/F$ the canonical quotient map$\}$. Let $e_0 = (1, 1, 1,...) \in l_\infty$ and $F_0 = [e_0]$. If $\xi \in l_p$, $\| \xi \|_p = 1$, choose N so that $| \xi_i | < \epsilon$ for all $i > N$. We can assume that $\xi_i \neq 0$ for all $i \leqslant N$. Let $| \xi_1 | = \max_{i \leqslant N} | \xi_i |$ and $| \xi_N | = \min_{i \leqslant N} | \xi_i |$ and $\delta_0 = 1/2(\xi_1 + \xi_N)$. Now

$$\| Q_F I \| = \sup_{\|\xi\|_p=1} \| Q_F \xi \| = \sup_{\|\xi\|_p=1} d(\xi, F).$$

For the above ξ,

$$d(\xi, F_0) \leqslant \sup_i | \xi_i - \delta_0 | \leqslant \max_{i \leqslant N} \{| \xi_i - \delta_0 |, | \delta_0 |\} + \epsilon,$$
$$\leqslant \max\{| \delta_0 | \, | \xi_1 - \delta_0 |\} + \epsilon,$$

by choice of δ_0 (since $| \xi_N - \delta_0 | = | \xi_1 - \delta_0 |$).

Let $g(x) = | x | + (1 - | x |^p)^{1/p}$. Then, the absolute maximum of g is $2/2^{1/p}$. Thus,

$$d(\xi, F_0) \leqslant \max\{| \delta_0 |, |\xi_1 - \delta_0 |\} + \epsilon = 1/2 \max\{| \xi_1 + \xi_N |, | \xi_1 - \xi_N |\} + \epsilon$$
$$\leqslant 1/2(| \xi_1 | + | \xi_N |) + \epsilon \leqslant 1/2 \cdot 2/2^{1/p} + \epsilon.$$

Since ξ was arbitrary, $\sup_{\|\xi\|_q \leqslant 1} d(\xi, F_0) \leqslant 1/2^{1/p}$. Equality occurs at $\xi_0 = (1/2^{1/p}, -1/2^{1/p}, 0, 0, 0,...) \in l_p$, (i.e., $d(\xi_0, F) = 1/2^{1/p}$.) Thus, $d_1(I) \leqslant \sup_{\|\xi\|^\cdot \leqslant 1} d(\xi, F_0) = 1/2^{1/p}$. To see that $d_n(I) = \sup_{\|\xi\|_p \leqslant 1} d(\xi, F_0)$ for each n, observe that

$$\inf\{\| Q_F I \|: \dim F = n, F \subset l^\infty\} = \inf\{\| Q_G I \|: \dim G = n$$

and G is equidistant from $(e_i)_{i=1}^{\infty} \subset l^\infty\} = \sup_{\|\xi\|_p=1} d(\xi, F_0) = 1/2^{1/p}$.

2.4. THEOREM. *If $1 \leqslant p \leqslant q < \infty$ and $I: l_p \to l_q$ is the natural injection, then $\alpha_k(I) = 1$ for all k.*

Proof. For the moment, let $I_r : l_r \to l_\infty$ denote the natural injection. Let $1 < p \leqslant q < +\infty$. Then, $I_p = I_q I$ and so

$$1 = \alpha_k(I_p) = \alpha_k(I_q I) \leqslant \| I_q \| \, \alpha_k(I) = \alpha_k(I)$$

for each k. That $\alpha_k(I) \leqslant 1$ is clear in this case.

If $p = 1$ and $\alpha_k(I) < 1$ for some k, choose $\epsilon > 0$ and an $A: l_1 \to l_q$ of rank at most k such that $\| I - A \| < 1 - \epsilon/2$. Let $B(A) = \{x \in A(l_1): \| x \|_q \leqslant \| A \|\}$. Then, $B(A)$ is relatively compact and $Ae_i = (a_{ij})_{j=1}^{\infty} \in B(A)$. Let $(y_i)_{i=1}^{m} \subset B(A)$ be an $\epsilon/10$-net for $B(A)$. If $y_i = (y_{ij})_{j=1}^{\infty}$, then there is an index j_0 such that $(\sum_{j=j_0}^{\infty} | y_{ij} |^q)^{1/q} < \epsilon/10$ for each i, $i = 1,..., m$. Also, for each i, there is an index $m(i)$ such that $(\sum_{j=1}^{\infty} | a_{ij} - y_{m(i)j} |^q)^{1/q} = \| Ae_i - y_{m(i)} \|_q < \epsilon/10$. Thus,

$$\left(\sum_{j=j_0}^{\infty} | a_{ij} |^q \right)^{1/q} \leqslant \left(\sum_{j=j_0}^{\infty} | a_{ij} - y_{m(i)j} |^q \right)^{1/q} + \left(\sum_{j=j_0}^{\infty} | y_{m(i)j} |^q \right)^{1/q}$$

$$< \epsilon/5 , \qquad \text{for each } i.$$

Hence, $| a_{ij} | < \epsilon/5$ for each i and each $j \geqslant j_0$. Since

$$1 - \epsilon/2 > \sup_i \|(I - A) \, e_i \| = \left(| 1 - a_{ii} |^q + \sum_{\substack{j=1 \\ j \neq i}}^{\infty} | a_{ij} |^q \right)^{1/q}$$

we have $| 1 - a_{ii} | < 1 - \epsilon/2$ for all i. Thus, for $i = j_0$, we obtain

$$1 - \epsilon/5 < 1 - a_{j_0 j_0} < 1 - \epsilon/2.$$

This absurdity shows that $\alpha_k(T) \geqslant 1$ for each k.

2.5. Corollary (to the proof). *For $1 \leqslant p \leqslant q < +\infty$ and for all n,*

$$d_n(I) = 1.$$

We now begin computing the approximation numbers of diagonal operators on l_p-spaces. Again, we must consider different cases depending on the relation between p and q. We first remark that Johnson in his thesis [15] has shown (replacing polytopes in 1.8 by suitable normal hulls in Kothe sequence spaces [19]) the following:

2.6. Theorem. *If $1 < q < p < \infty$ and $T: l_p \to l_q$ is a diagonal operator determined by (λ_n), then*

$$\alpha_n(T) = \left(\sum_{i=k+1}^{\infty} | \lambda_i |^r \right)^{1/r}, \qquad \text{where } \frac{1}{r} + \frac{1}{p} = \frac{1}{q}.$$

For a diagonal mapping T from l_p to l_p, $T \sim (\lambda_n)$, it is well known (see, e.g., [28, 30]) that $d_k(T) = \alpha_k(T) = \lambda_{k+1}$.

Unfortunately, we have been unable to compute exactly the approximation numbers of a diagonal from l^p to l^q, $1 \leqslant p < q \leqslant \infty$. We give asymptotic estimates below that, in a sense, are the best possible. We first state an elementary lemma whose proof is immediate using the simplex method.

2.7. Lemma. *Let* (λ_n) *be a decreasing sequence and* $1 \leqslant r < \infty$. *If* $\theta_n = \sup\{\mu_{n+1} : \Sigma \mu_i{}^r = 1, \lambda_i \mu_i \geqslant \lambda_{i+1}\mu_{i+1}\}$, *then*

$$\theta_n = \lambda_{n+1}^{-r} \left(\sum_{i=1}^{n+1} \lambda_i^{-r} \right)^{-1/r}.$$

2.8. Theorem. *If* $T: l_1 \to l_\infty$ *is a diagonal,* $T \sim (\lambda_i)_{i=1}^\infty$, *then* $(\sum_1^{k+1} \lambda_i^{-1})^{-1} \leqslant \alpha_k(T) \leqslant \min(\lambda_k/2, \lambda_{k+1})$ *for each* k.

Proof. Let $S: l_1 \to l_1$, $S \sim (\lambda_i)$. Then, we have

$$\begin{tikzcd} l_1 \arrow[rr, "T"] \arrow[dr, "S"'] & & l_\infty \\ & l_1 \arrow[ur, "I"'] & \end{tikzcd}$$

Thus, $\alpha_k(T) = \alpha_k(IS) \leqslant \alpha_1(I)\, \alpha_{k-1}(S) = \frac{1}{2}\lambda_k$.

On the other hand, if $S: l_\infty \to l_1$ corresponds to (μ_n), where (μ_n) satisfies the hypotheses of 2.7, then we have

$$l_1 \xrightarrow{T} l_\infty \xrightarrow{S} l_1 \qquad \text{and} \qquad \mu_{k+1}\lambda_{k+1} = \alpha_n(ST) \leqslant \alpha_n(T) \| S \| = \alpha_k(T).$$

Thus, by 2.7, $\alpha_k(T) \geqslant (\sum_{i=1}^{k+1} \lambda_i^{-1})^{-1}$.

2.9. Theorem. *If* $1 \leqslant p < q \leqslant \infty$ *and* $T \sim (\lambda_n)$, *then* $(\sum_{i=1}^{k+1} \lambda_i^{-r})^{-1/r} \leqslant \alpha_k(T) \leqslant \lambda_{k+1}$ *for each* k. *Here,* $1/r + 1/q = 1/p$.

Proof. The left-hand inequality follows from 2.7 as above. The right-hand inequality is immediate by considering the kth truncation of T.

Let us again remark that the above estimates are the best asymptotic estimates possible. Indeed, the injection $l_1 \to l_\infty$ shows that the right-hand side can be attained. On the other hand, it is easy to see that if I_n denotes the natural injection of $l_1(n) \to l_\infty(n)$, then $\alpha_{n-1}(I_n) = 1/n$. Let $\lambda_i = 1$, $i \leqslant n$, $\lambda_i = 0$ for $i > n$ and $T = l_1 \to l_\infty$, $T \sim (\lambda_i)$. Then $\alpha_{n-1}(T) = 1/n$, which is the left-hand side of 2.8.

Thus, in general, $\alpha_k(T) \neq \lambda_{k+1}$. Indeed, we will now calculate the exact value of $\alpha_k(T) = d_k(T)$ for $T: l_1 \to l_2$, a diagonal corresponding to a decreasing null sequence. For this, we need two lemmas. The first is combinatorial in nature.

2.10. LEMMA. *For fixed k and $n \geqslant k+1$, if $(\lambda_i)_{i=1}^{n+1}$ are scalars satisfying $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n > \lambda_{n+1} = 0$, then there is a unique integer i, $k+1 \leqslant i \leqslant n$ such that*

$$\lambda_i > \left((i-k)\Big/\sum_{j=1}^{i} \lambda_j^{-2}\right)^{1/2} > \lambda_{i+1}.$$

Proof. If no such integer exists, then, since $(n-k)/\sum_{j=1}^{n} \lambda_j^{-2} > 0 = \lambda_{n+1}$, we must have

$$\lambda_n^{\,2} < (n-k)\Big/\sum_{j=1}^{n} \lambda_j^{-2}, \qquad \text{or} \qquad 1 + \lambda_n^{\,2} \sum_{j=1}^{n-1} \lambda_j^{-2} < n-k.$$

Since the lemma is assumed not to hold, we must have

$$\lambda_{n-1}^2 < ((n-1)-k)\Big/\sum_{j=1}^{n-1} \lambda_j^{-2}$$

and as above, this leads to

$$\lambda_{n-1}^2 < ((n-2)-k)\Big/\sum_{j=1}^{n-2} \lambda_j^{-2}.$$

Continuing in this manner, we eventually obtain

$$\lambda_{k+1}^2 < ((k+1)-k)\Big/\sum_{j=1}^{k+1} \lambda_j^{-2}.$$

or $1 + \lambda_{k+1}^2 \sum_{j=1}^{k} \lambda_j^{-2} < 1$. Thus, such an i exists. Now, suppose that the lemma holds for some integer s, $k+1 \leqslant s \leqslant n$ with, say

$$\lambda_i \geqslant \left((i-k)\Big/\sum_{j=1}^{2} \lambda_j^{-2}\right)^{1/2} \geqslant \lambda_{i+1} \geqslant \lambda_{i+2} \geqslant \cdots \geqslant \lambda_s$$

$$\geqslant (s-k)\Big/\sum_{j=1}^{s} \lambda_j^{-2} > \lambda_{s+1}.$$

Thus,

$$(*) \qquad (i-k)/s-k > \sum_{j=1}^{i} \lambda_j^{-2}\Big/\sum_{j=1}^{s} \lambda_j^{-2}.$$

But

$$\lambda_{i+1}^2 \geqslant \lambda_{i+2}^2 \geqslant \cdots \geqslant \lambda_s^2 \geqslant (s-k)\Big/\sum_{j=1}^{s} \lambda_j^{-2}$$

implies

$$\sum_{j=i+1}^{s} \lambda_j^{-2} \leqslant ((s-i)/(s-k)) \sum_{j=1}^{s} \lambda_j^{-2}$$

and so $-\sum_{j=1}^{i} \lambda_j^{-2} \leqslant (k-i)/(s-k)) \sum_{j=1}^{s} \lambda_j^{-2}$. Then, from (*) above, we obtain

$$\sum_{j=1}^{i} \lambda_j^{-2} \geqslant ((i-k)/(s-k)) \sum_{j=1}^{s} \lambda_j^{-2} > \sum_{j=1}^{i} \lambda_j^{-2}.$$

Thus, the integer i satisfying the lemma is unique.

2.11. Lemma. *Let $T_n : l_1(n) \to l_2(n)$ be defined by $T_n e_i = \lambda_i e_i$ for each i with $(\lambda_i)_{i=1}^n$ as in Lemma 2.10. Then*

$$\alpha_k(T_n) = \inf \left\{ \max_{i \leqslant n} \lambda_i \left[1 - \sum_{j=1}^{k} x_{ji}^2\right]^{1/2} : x_m = (x_{m_i}) \in l_2(n)\right.$$

with $(x_m, x_n) = \delta_{mn}$ for $1 \leqslant m, n \leqslant k\}$. Here $(\ ,\ )$ denotes the usual inner product on $l_2(n)$.

Proof. Let $\mathfrak{A}_k$ denote the right-hand side of the above expression. Let $\epsilon > 0$ and $A \in A_k(l_1(n), l_2(n))$ for $k \leqslant n$. If $H = A(l_1(n))$, then $H = [x_i]$, where $(x_i, x_j) = \delta_{ij}$, $i \leqslant k$. Thus,

$$\| T_n - A \| = \max_{i \leqslant n} \|(T_n - A)\, e_i\| = \max_{i \leqslant n} \| \lambda_i e_i - A e_i \|.$$

If $x_i = (x_{ij})_{j=1}^n$, then $d(\lambda_i e_i, H) = \lambda_i [1 - \sum_{j=1}^{k} x_{ji}]^{1/2}$ for each i and so

$$\| T_n - A \| \geqslant \max_{i \leqslant n} \lambda_i \left[1 - \sum_{j=1}^{k} x_{ji}^2\right], \qquad \text{i.e.,}$$

$$\alpha_n(T) \geqslant \mathfrak{A}_k, \qquad \text{for each } k \leqslant n.$$

Now, let $(x_i) \subset l_2(n)$ be such that $(x_i, x_j) = \delta_{ij}$, $1 \leqslant i, j \leqslant k$, where $x_i = (x_{ij})_{j=1}^n$. Let $H = [x_i]$ and let $P: l_2(n) \to H$ be the projection $Pe_i = \sum_{j=1}^{k} (x_j, e_i)\, x_j$. Then,

$$d(\lambda_i e_i, H) = \| \lambda_i e_i - P\lambda_i e_i \| = \lambda_i \left[1 - \sum_{j=1}^{n} x_{ji}^2\right]^{1/2} \qquad \text{for each } i,\ 1 \leqslant i \leqslant n.$$

Let $I: l_1(n) \to l_2(n)$ be the natural injection and define $S: l_2(n) \to l_2(n)$ by $Se_i = \lambda_i e_i$, $1 \leqslant i \leqslant n$. Thus, PSI: $l_1(n)$ is of rank k and

$$\begin{aligned}
\| T_n - PSI \| &= \max_{j \leqslant n} \|(T_n - PSI)\, e_j \|_2 , \\
&= \max_{j \leqslant n} \| \lambda_j e_j - P\lambda_j e_j \|_2 , \\
&= \max_{j \leqslant n} d(\lambda_j e_j , H), \\
&= \max_{j \leqslant n} \lambda_j \left[1 - \sum_{m=1}^{k} x_{mj}^2 \right]^{1/2},
\end{aligned}$$

and so $\alpha_k(T_n) \leqslant \mathfrak{A}_k$.

Observe that $\mathfrak{A}_k = \inf\{\max_{i\leqslant n} \lambda_i[1 - A_i]^{1/2}: A_i \in [0, 1]$ for $i \leqslant n$ and $\sum_{i=1}^n A_i = 1\} = \inf\{\max_{i\leqslant n} \lambda_i B_i^{1/2}: B_i \in [0, 1]$ and $\sum_{i=1}^n B_i = n - k\} = \inf_{k\leqslant r\leqslant n} \{\max[\lambda_1 B_1^{1/2},\ldots, \lambda_r B_r^{1/2}, \lambda_{r+1}],\ B_i \in [0, 1]$ with $\sum_{i=1}^r B_i = r - k\}$.

2.12. THEOREM. *For fixed integers k, n, $k + 1 \leqslant n$, there is a unique integer $r(k, n)$, $k + 1 \leqslant r(k, n) \leqslant n$ such that*

$$\alpha_k(T_n) = \left[(r(k, n) - k) \Big/ \sum_{j=1}^{r(k,n)} \lambda_j^{-2} \right]^{1/2},$$

where T_n is as in Lemma 2.11.

Indeed, choose $r = r(k, n)$ as in Lemma 2.10. Using the remark following the proof of 2.11, Theorem 2.12 follows. Now, observe that if T is a diagonal map $l_1 \to l_2$ corresponding to a monotonically decreasing null sequence, then $T = \lim_n T_n$, where $T_n = TP_n$ and $P_n : l_1 \to l_1(n)$ is the canonical projection. Since $| \alpha_k(T) - \alpha_k(S)| \leqslant \alpha_k(T - S)$ for any T and S, we have $\alpha_k(T) = \lim_{n\to\infty} \alpha_k(T_n)$ for each k. Using this remark and Theorem 2.12, one can compute $d_k(T) = \alpha_k(T)$ for such T.

2.13. EXAMPLE. Let $T: l_1 \to l_2$ be given by $Te_i = i^{-1/2} e_i$. Using Theorem 2.12, it is easy to show that $\alpha_n(T) = 1/(2n + 1)^{1/2}$ for each n.

Since the approximation numbers and Kolmogoroff diameters are homogeneous, we can summarize (and generalize) our results 2.1–2.5 as follows:

2.14. THEOREM. *Suppose that (λ_n) is an increasing sequence with limit λ. Then*:

(i) *If $T: l_1 \to l_\infty$, $T \sim (\lambda_n)$, $d_n(T) = \alpha_n(T) = \lambda/2$ for all n; and if $T: l_1 \to c_0$, $d_n(T) = \alpha_n(T) = \lambda$ for all n;*

(ii) *if* $T: l_p \to l_\infty$, $p > 1$, $\alpha_n(T) = \lambda$ *for all* n *and* $d_n(T) = \lambda \cdot 2^{-1/p}$ *for all* n;

(iii) *if* $1 \leqslant p \leqslant q < \infty$ *and* $T: l_p \to l_q$, *then* $d_n(T) = \alpha_n(T) = \lambda$ *for all* n.

We end this section with a few remarks concerning the relationships between p-absolutely summing, nuclear, and type l_p-operators. Pietsch [36] has shown that the composition of three 1-absolutely summing operators is of type l_1. Example 1.5 shows that this result of Pietsch is the best possible; for, the operator of Example 1.5 is the composition of two 1-absolutely summing operators, but not of type l_1.

If T is of type l_p, $p > 1$, not much can be said about T other than the obvious fact that T is compact. Indeed, if $p > 2$, choose $(\lambda_n) \in l_p \backslash l_2$ with $|\lambda_i| \geqslant |\lambda_{i+1}|$. Construct the diagonal $T: l_2 \to l_2$ corresponding to (λ_n). Then, T is of type l_p and yet T is not q-absolutely summing for any value of q. Indeed, if T were q-absolutely summing for some value of q, then T would be Hilbert–Schmidt, hence, of type l_2 [36]. But $\alpha_k(T) = |\lambda_{k+1}|$ and $(\lambda_n) \notin l_2$.

3. H-OPERATORS AND ϵ_p-FINITE BASES

In Section 1, we considered the relationships between the approximation numbers and Kolmogoroff diameters of arbitrary operators between Banach spaces. Of course, as seen in Section 2, by placing restrictions on the operators better estimates can be given. Indeed, when one considers $T \in \mathscr{L}(E, E)$, the techniques of spectral theory are available and theorems, analogous to results on Hilbert space [8, 26, 27] can be obtained. Markus [26] has proved the following beautiful result relating the eigenvalues, Kolmogoroff diameters and approximation numbers of H-operators. An operator T acting on a Banach space E is an *H-operator* if its spectrum is real and its resolvent satisfies

$$\|(T - \lambda I)^{-1}\| \leqslant C\,|\operatorname{Im} \lambda|^{-1} \qquad (\operatorname{Im} \lambda \neq 0).$$

Here, of course, C is independent of the points of the resolvent. We point out that an operator on a Hilbert space is an H-operator with constant $C = 1$ if and only if it is a self-adjoint operator.

Now, we state the result of Marcus.

3.1. THEOREM. *If T is a compact H-operator on a Banach space E, then for each n*

$$d_n(T) \leqslant \alpha_n(T) \leqslant 2(2^{1/2})\,|\lambda_{n+1}(T)| \leqslant 8C(C + 1)\, d_n(T).$$

Here, $(\lambda_n(T))$ denotes the sequence of eigenvalues of T numbered in order of decreasing modulus and taking into account their multiplicity and C is the constant occurring in the definition above.

In a certain sense, the diagonal elements (λ_n) of a diagonal basis map $e_n \to \lambda_n u_n$ share some of the properties of eigenvalues. Thus, we are led to a generalization of Theorem 3.1 to diagonal mappings between certain types of bases.

We say that a basis (x_i) for a Banach space E is *ϵ_p-finite* provided $\epsilon_p(x_n) < +\infty$ (see the Introduction). In the terminology of Grothendieck [11], an ϵ_p-finite basis is a basis that is *weakly p-summable*.

It is clear from Holder's inequality that if $1/p + 1/p' = 1$ and $(f_i) \subset E'$, $(y_i) \in F$, $\epsilon_p(f_i) < +\infty$ and $\epsilon_{p'}(y_i) < +\infty$, then $\epsilon_1(f_i \otimes y_i) < +\infty$.

3.2. PROPOSITION. *If (f_i) and (y_i) are as above and (λ_n) is a null sequence, then $T = \sum_{i=1}^{\infty} \lambda_i f_i \otimes y_i$ defines a compact operator from E to F.*

Proof. For any N, $\| \sum_{i=N}^{\infty} \lambda_i f_i \otimes y_i \| \leqslant \max_{i \geqslant N} |\lambda_i| \, \epsilon_1(f_i \otimes y_i)$ and this latter expression tends to 0 as N tends to ∞. Thus, $T \in \mathscr{F}(E, F)$.

Now, we can prove the main result of this section.

3.3. THEOREM. *Let T be a diagonal basis mapping $e_n \to \lambda_n u_n$, where (e_n, f_n) and (u_n, g_n) are bases for E and F, respectively, and (λ_n) is a monotonically decreasing null sequence $\lambda_n \neq 0$. Suppose that $\epsilon_1(g_n \otimes e_n)$ and $\epsilon_1(f_n \otimes u_n)$ are finite. Then, we have*

$$[\epsilon_1(g_n \otimes e_n)]^{-1} \lambda_{n+1} \leqslant d_n(T) \leqslant \alpha_n(T) \leqslant [\epsilon_1(f_n \otimes u_n)] \lambda_{n+1} .$$

Proof. For each integer n, let T_n denote the restriction on T to $[e_1, ..., e_n]$. Then, T_n is invertible and $T_n^{-1} u_i = \lambda_i^{-1} e_i$ for $i = 1, 2, ..., n$. Thus,

$$\| T_n^{-1} \| = \sup \left\{ \left\| \sum_{i=1}^{n} \lambda_i^{-1} g_i(y) e_i \right\| : \| y \| = 1, y \in [u_1 \cdots u_n] \right\}$$

$$\leqslant \lambda_n^{-1} \epsilon_1(g_i \otimes e_i) \leqslant (\lambda_{n+1}^{-1} \epsilon_1(g_i \otimes e_i).$$

Now, let F_n be an arbitrary n-dimensional subspace in F. By the remark after the proof of the Krein–Milman–Krasnoselskii theorem, there is a $y \neq 0$ in $[u_1, ..., u_{n+1}]$ such that $d(y, F_n) = \| y \|$. Let $x = T_n^{-1} y$ and suppose, without loss of generality, that $\| x \| = 1$. Then,

$$d(y, F_n) = \| y \| \geqslant \| T_n^{-1} \|^{-1} \geqslant \lambda_{n+1} [\epsilon_1(g_i \otimes e_i)]^{-1}.$$

Also,

$$\alpha_n(T) \leqslant \left\| \sum_{i=n+1}^{\infty} \lambda_i f_i \otimes u_i \right\| = \lambda_{n+1} \left\| \sum_{i=n+1}^{\infty} \lambda_i/\lambda_{n+1} f_i \otimes u_i \right\|$$

$$\leqslant \lambda_{n+1} \epsilon_1(f_i \otimes u_i).$$

At fitst glance, Theorem 3.3 appears hard to apply. Indeed, if (x_n, f_n) is a basis for a Banach space E with $0 < \inf_n \| x_n \| \leqslant \sup_n \| x_n \| < +\infty$ and $\epsilon_2(x_n) < +\infty$, $\epsilon_2(f_n) < +\infty$, then (x_n) is equivalent to the unit vector basis of l_2, i.e., E is isomorphic to l_2 under an isomorphism. $\varphi(x_n) = e_n$. However, by relaxing the conditions of 3.3 slightly, we obtain a result valid for certain subspaces of arbitrary Banach spaces.

From the proof in [31], it is not difficult to obtain the following result.

3.4. THEOREM. *Let E be an infinite-dimensional Banach space, $\epsilon > 0$, and (a_i) a null sequence with $0 < a_i < 1$ for each i. Then, there is a basic sequence (x_n) in E with coefficient functionals (f_n) in $[x_n]^*$ such that*

$$1 - \epsilon \leqslant \min_n (\| x_n \|, \| f_n \|) \leqslant \max_n (\| x_n \|, \| f_n \|) < 1 + \epsilon,$$

with $\epsilon_2(a_i x_i) = \epsilon_2(a_i f_i) = 1$.

To see how 3.4 applies to 3.3, consider the following: Let (λ_n) be a monotinically decreasing null sequence with $0 < \lambda_n < 1$. For $\epsilon > 0$, choose $\delta_n > 0$ such that $\lambda_n^{1-2\delta_n} < (1+\epsilon)\lambda_n$ (possible since $\lambda_n^{-1} > 1$) and such that $\lim_n \delta_n \mid \log \lambda_n \mid = 0$. Let $a_n = \lambda_n^{\delta_n/2}$ and $b_n = \lambda_n^{1-\delta_n}$. Then, $(a_n), (b_n) \in c_0$ by the choice of δ_n and $0 < a_n < 1$. Thus, if

$$T = \sum_{n=1}^{\infty} b_n f_n \otimes y_n,$$

where (x_n, f_n) and (y_n, g_n) are basic sequences satisfying 3.4, then $T = \sum_{n=1}^{\infty} b_n a_n^{-2}(a_n f_n) \otimes (a_n y_n)$ and so (as in the proof of Theorem 3.3) we obtain

$$\lambda_{n+1} = a_{n+1}^2 b_{n+1} \leqslant d_n(T) \leqslant \alpha_n(T) \leqslant b_{n+1} a_{n+1}^{-2} \leqslant (1+\epsilon)\lambda_{n+1}.$$

Thus, it is possible to construct an operator T on certain infinite-dimensional subspaces of arbitrary Banach spaces E and F analogous to the compact H-operators, in the sense of the conclusion of Theorem 3.1. We will make use of this result and the technique of [31] in the next section.

4. Bernstein Pairs

Motivation for this section is the following classical result of Bernstein (see, e.g., [4]).

4.1. Theorem. *Let E be a Banach space and let (x_n) be a linearly independent sequence of elements of E. Given an arbitrary positive monotonically decreasing null sequence (b_n), there is an $x_0 \in E$ such that $d(x_0, [x_1, \ldots, x_n]) = b_n$ for each n.*

This leads to the following definition.

4.2. Definition. Two Banach spaces E and F are said to form a *Bernstein pair* if for any positive monotonic null sequence (b_n) there is a $T \in \mathscr{L}(E, F)$ and a constant M, depending only on T and (b_n), such that $b_n \leqslant \alpha_n(T) \leqslant Mb_n$ for all n. We say that $(\alpha_n(T))$ is equivalent to (β_n). We write $\langle E, F\rangle$ is a *Bernstein pair.*

Some remarks concerning Definition 4.2 are in order.

In general, if $\langle E, F\rangle$ is a Bernstein pair, there are no equivalent norms on E and F such that $\tilde{\alpha}_n(T) = \beta_n$. (Here, $\tilde{\alpha}_n(T)$ denotes the nth approximation number of T with respect to given norms.)

Indeed, suppose that $|||\ |||$ and $|\ |$ are equivalent norms on E and F, respectively, say

$$L_2 |||x||| \leqslant \|x\| \leqslant L_1 |||x||| \qquad \text{and} \qquad K_2 |y| \leqslant \|y\| \leqslant K_1 |y|$$

for all $x \in E$, $y \in F$. Then,

$$\sup_{|||x|||} |(T - A)x| \leqslant L_1 K_2^{-1} \sup_{\|x\| \leqslant 1} \|(T - A)x\|,$$

i.e., $\tilde{\alpha}_n(T) \leqslant L_1 K_2^{-1} \alpha_n(T)$. Similarly, $\tilde{\alpha}_n(T) \geqslant L_2 K_1^{-1} \alpha_n(T)$, for all $T \in \mathscr{L}(E, F)$. Since $\langle E, F\rangle$ is a Bernstein pair, there is a $M > 0$ and T such that $b_n \leqslant \alpha_n(T) \leqslant Mb_n$, where (b_n) is a preassigned null sequence. Thus, the inequalities yield

$$K_2 \leqslant K_1, \qquad L_2 \leqslant L_1, \qquad L_2 K_1^{-1} \geqslant 1, \qquad L_1 K_2^{-1} \leqslant M^{-1}$$

and so $K_1 \leqslant M^{-1} K_2$ or $K_1 \leqslant M^{-1} K_1$, i.e., $M \leqslant 1$. From the definition $M \geqslant 1$, thus, $M = 1$.

4.3. *Remark.* It is clear from 4.1 that by choosing *any* sequence of linearly independent rank one operators in $\mathscr{L}(E, F)$, E and F arbitrary and any null sequence (b_n) as above, that there is an infinite rank T_0 such that

$$\alpha_n(T_0) \leqslant b_n .$$

Thus, it is conceivable that *any* two infinite-dimensional Banach space E and F form a Bernstein pair. It would be of considerable interest to know, e.g., if $\langle E, E\rangle$ is a Bernstein pair for any infinite-dimensional Banach space E. Of special interest in this case would be to assertain when the operator T could be chosen to be an H-operator (it would be, of course, compact).

4.4. *Remark.* Let E and F be arbitrary infinite-dimensional Banach spaces. Then, there are infinite-dimensional spaces E_0 and F_0 in E and F, respectively, such that $\langle E_0, F_0\rangle$ is a Bernstein pair. Moreover, the operator T satisfying the definition for $\langle E_0, E_0\rangle$ can be chosen to be an H-operator.

The proof of 4.5 is immediate from 3.4. Indeed, for $\epsilon > 0$, the spaces E_0 and F_0 can be chosen to have Schauder bases and the M accuring in Definition 4.2 can be taken to be $(1 + \epsilon)$. The operator $T \in \mathscr{L}(E_0, E_0)$ constructed in 3.4 will be an H-operator [26].

4.5. *Remark.* If $E = E_1 \oplus E_2$ and $F = F_1 \oplus F_2$ and $\langle E_i, F_i\rangle$ is a Bernstein pair for $i = 1$ or 2 then $\langle E, F\rangle$ is a Bernstein pair.

Proof. Without loss of generality, suppose that $\langle E_1, F_1\rangle$ is a Bernstein pair. Let $T \in \mathscr{L}(E_1, F_1)$ be such that $\beta_n \leqslant \alpha_n(T) \leqslant M\beta_n$.

Let $\tilde{T} = TP_1 \in \mathscr{L}(E, F)$, where P_1 is the projection from E onto E_1. Also, let Q_1 be the projection of F onto F_1. Then, $\alpha_n(\tilde{T}) = \alpha_n(TP_1) \leqslant \| P_1 \| \alpha_n(T)$.

For $\epsilon > 0$ let A be an operator of rank at most n, such that $\| \tilde{T} - A \| \leqslant \alpha_n(\tilde{T}) + \epsilon$ and let $B = Q_1A_1$, where A_1 is the restriction of A to E_1. Then,

$$\| T - B \| = \sup_{\substack{\|x\|\leqslant 1\\ x\in E_1}} \|(\tilde{T} - Q_1A_1)(x)\| = \sup_{\substack{\|x\|\leqslant 1\\ x\in E_1}} \| Q_1\tilde{T} - Q_1A_1(x)\|$$

$$\leqslant \| Q_1 \| \, \| \tilde{T} - A \| \leqslant \| Q_1 \| \, (\alpha_n(\tilde{T}) + \epsilon).$$

But $\alpha_n(T) \leqslant \| T - B \|$. Thus,

$$\| Q_1 \|^{-1} \beta_n \leqslant \| Q_1 \|^{-1} \alpha_n(T) \leqslant \alpha_n(\tilde{T}) \leqslant \| P_1 \| \alpha_n(T) \leqslant \| P_1 \| M\beta_n,$$

i.e., $\langle E, F\rangle$ is a Bernstein pair.

4.6. *Remark.* If E has a Schauder basis, then $\langle E, E\rangle$ is a Bernstein pair.

Proof. If $\lambda_n \searrow 0$, then the operator $T = \sum_{n=1}^{\infty} \lambda_n f_n \oplus x_n$, where $\langle x_n, f_n\rangle$ is a basis for E, is a compact H-operator [26], with the sequence (λ_n) as eigenvalues. The result follows from Theorem 4.1.

In particular, if E is a separable $\mathscr{L}_p$-space $1 \leqslant p \leqslant \infty$, then $\langle E, E\rangle$ is a Bernstein pair. Indeed, every separable $\mathscr{L}_p$-space has a basis [39].

Next, we consider the relationship between the approximation numbers of an operator T and its adjoint T'. It is obvious from the definition that

$\alpha_n(T') \leqslant \alpha_n(T)$. By considering the injection $l_1 \to c_0$ we see that strict inequality can occur. Now, we show that, for certain operators, equality is always obtained.

4.7. THEOREM. *Let E and F be Banach spaces and let $T \in \mathscr{F}(E, F)$. Then, $\alpha_n(T) = \alpha_n(T')$ for every n. (In particular, if F has the approximation property, equality holds for all compact operators.*

(Hutton has recently observed that 4.7 is valid for arbitrary compact operators.)

Proof. Let $S\colon E'' \to F$ be a finite-rank operator and let $j\colon F \to F''$ be the canonical mapping. Let $\beta_k(S) = \inf\{\| S - A \|\colon A\colon E'' \to F,\ \text{rank } A \leqslant k\}$. Choose, for $\epsilon > 0$, $A_k : E'' \to F''$ such that $\| jS - A_k \|_{F''} < \alpha_k(jS) + \epsilon$. Let G be the subspace spanned by $jS(E'') \cup A_k(E'')$. By the principle of local reflexivity there is an operator $\varphi\colon G \to F$ such that $\| \varphi \| = 1, \| \varphi^{-1} \| \leqslant 1 + \epsilon$ and φ restricted to $G \cap jF$ is the identity. Consider $\varphi A_k : E'' \to F$. If $\| x'' \| \leqslant 1$, we have

$$\|(S - \varphi A_k)\, x'' \|_F = \| \varphi jSx'' - \varphi A_k x'' \|_F \leqslant \| jSx'' - A_k x'' \|_{F''}$$

and so

$$\| S - \varphi A_k \| \leqslant \alpha_k(jS) + \epsilon, \qquad \text{i.e., } \beta_k(S) \leqslant \alpha_k(jS).$$

Since $T \in \mathscr{F}(E, F)$, $T''\colon E'' \to jF$ and there are finite-rank operators $S_n \in \mathscr{L}(E'', jF)$ such that $\lim_n \| S_n - T'' \| = 0$. For $\epsilon > 0$, choose N so that $\| T'' - S_n \| < \epsilon/2$ if $n \geqslant N$. Since $|\beta_k(T'') - \beta_k(S_n)| \leqslant \| T'' - S_n \|$, we obtain

$$\beta_k(j^{-1}T'') < \beta_k(j^{-1}S_n) + \epsilon/2, \qquad \text{for } n \geqslant N.$$

And from the above, we obtain

$$\beta_k(T'') \leqslant \beta_k(j^{-1}T'') < \beta_k(j^{-1}S_n) + \epsilon/2 < \alpha_k(S_n) + \epsilon/2.$$

It follows that

$$\beta_k(T'') \leqslant \alpha_k(T'').$$

Clearly, $\alpha_k(T) \leqslant \beta_k(T'')$. Thus,

$$\alpha_k(T) \leqslant \beta_k(T'') \leqslant \alpha_k(T'') \leqslant \alpha_k(T') \leqslant \alpha_k(T).$$

As an immediate corollary to Theorem 4.7, we obtain the following fact concerning Bernstein pairs.

4.8. COROLLARY. *Suppose that $\langle F', E' \rangle$ is a Bernstein pair, then, $\langle E'', F'' \rangle$ is a Bernstein pair. If $\langle E, F \rangle$ is a Bernstein pair so is $\langle F', E' \rangle$.*

Our goal is to show that the "classical" Banach spaces form Bernstein pairs. More precisely, we wish to show that if E is an $\mathscr{L}_p$-space and F and $\mathscr{L}_q$-space, in the sense of Lindenstrauss and Pelczynski, then $\langle E, F\rangle$ is a Bernstein pair. The idea of the proof of this fact is to reduce the problem to the l_p-spaces. It will become apparent that the main obstruction to doing this is the lack of suitable structure theorems for $\mathscr{L}_\infty$-spaces.

4.9. THEOREM. *Let E be l_∞ or c_0 and F one of the spaces l_p, $1 \leqslant p < \infty$. Then $\langle E, F\rangle$ and $\langle F', E'\rangle$ are Bernstein pairs.*

Proof. The result is immediate from Theorems 1.8 and 4.7. If $\beta_n \searrow 0$, let $\lambda_n = (\beta_n{}^p - \beta_{n+1}^p)^{1/p}$ and let T be the diagonal mapping corresponding to (λ_n).

The above result is also valid of the roles of E and F are interchanged. To prove this, we need the following lemma. We recall that a basis (x_n, f_n) is shrinking if (f_n) is a basis for E.

4.1. LEMMA. *Suppose that E is reflexive or E' is separable and F has a shrinking Schauder basis. Then, if $\langle E, F''\rangle$ is a Bernstein pair, so is $\langle E, F\rangle$.*

Proof. Let $\beta_n \searrow 0$ be given and let $T: E \to F''$ be such that $(\alpha_n(T))$ is equivalent to (β_n). Also, let S_n denote the nth partial sum operator associated with some shrinking basis for F. By the principle of local reflexivity there are mappings

$$Q_n: S_n^{**}(TE) \to F, \qquad \text{with} \quad \| Q_n \| = 1,$$

$\| Q_n^{-1} \| \leqslant 2$ for all n with Q_n the identity on $S_n^{**}(TE) \cap jF$. Since E is reflexive, the sequence $([Q_n S_n^{**} T]^*)$ clusters in the weak operator topology to a bounded operator $S^*: F' \to E'$. We claim that $(\alpha_k(S)) \sim (\alpha_k(T))$. To see this, observe that since each α_n is pointwise continuous we have $\alpha_n(S) \leqslant \lim_{m\to\infty} \sup \alpha_n(Q_m S_m^{**} T) \leqslant K\alpha_n(T)$. where $K = \sup_m \| S_m \|$. For the opposite inequality, for $\epsilon > 0$ choose $B: E \to F$ such that rank $B \leqslant n$ and $\alpha_n(S) > \| S - B \| - \epsilon$. Then,

$$\begin{aligned}
\alpha_n(S) + \epsilon > \| S - B \| &= \lim_{m\to\infty} \inf \sup_{\substack{x\in E\\ \|x\|=1}} \| Q_n S_n^{**} Tx - Bx \| \\
&\geqslant \lim_{m\to\infty} \inf \sup_{\substack{x\in E\\ \|x\|=1}} \| Q_n S_n^{**} Tx - Q_n S_n^{**} jBx \| \\
&\geqslant \tfrac{1}{2} \lim_{m\to\infty} \inf \| S_n^{**} T - S_n^{**} T - S_n^{**} jB \| \\
&\geqslant \tfrac{1}{2} \| T - jB \| \geqslant \tfrac{1}{2}\alpha_n(T).
\end{aligned}$$

The proof for E' separable is essentially the same since, in this case, the unit ball of E' is w^*-sequentially compact. (In the preceding, j denotes the canonical map from $F \to F^{**}$. Since the basis is shrinking, $\| S_n^{**} T \| \to \| T \|$.

Indeed, $S_n^* \to I$, the identity, in the strong operator topology and the result follows.)

4.11. COROLLARY. *For $1 < p < \infty$, $\langle l_p, c_0\rangle$ is a Bernstein pair. Thus, for this range of p, $\langle l_1, l_p\rangle$ is a Bernstein pair.*

Proof. The space c_0 has a shrinking basis and so the result is immediate from 4.10.

Clearly, $\langle l_p, l_\infty\rangle$ is a Bernstein pair. Indeed l_∞ contains l_p isometrically and the natural diagonal suffices. That $\langle l_1, l_p\rangle$ is a Bernstein pair for $1 < p < \infty$ now follows from 4.7.

The extreme case $\langle l_1, c_0\rangle$ requires a separate argument.

4.12. THEOREM. *The couple $\langle l_1, c_0\rangle$ forms a Bernstein pair.*

Proof. Write $l_1 = (\bigoplus l_1(n))_1$ and find $u_i^{(n)} \in c_0$ such that for each n, $[u^{(n)} \cdots u_1^{(n)}] = G_n$ is isometric to $l_1(n)$ and define, for $\beta_n \searrow 0$,

$$T\colon (\bigoplus l_1(n))_1 \to (\bigoplus G_n)_0 \subset c_0, \qquad \text{by}$$

$$T(\xi_i) = T((\xi_i^{(n)})) = \sum_n \left(\sum_{i=1}^{n} \beta_i^{(n)} \xi_i^{(n)} u_i{}^n\right).$$

Here, $(\xi_i) \in l_1$ and (β_i) are "blocked" according to the above decomposition. If n_k is such that $n_k < k \leqslant n_{k+1}$, then

$$\alpha_k(T) \leqslant \sup_{\|(\xi_n)\|\leqslant 1} \left\| \sum_{k+1}^{n_{k+1}} \beta_i^{(n_k)} \xi_i^{(n_k)} u_i^{(n_k)} + \sum_{n=n_{k+1}+1}^{\infty} \sum_{i=1}^{n} \beta_i^{(n)} \xi_i^{(n)} u_i^{(n)} \right\|$$

$$\leqslant \beta_{k+1} \sup_{\|(\xi_n)\|\leqslant 1} \sum_{i=k+1}^{\infty} |(\beta_i/\beta_{k+1})\, \xi_i\,| = \beta_{k+1}\,.$$

For the opposite inequality, if $A = l_1 \to c_0$ has rank $\leqslant k$, then there are $\xi_1 \cdots \xi_{k+1}$ such that $\sum_1^{k+1} |\,\xi_i\,| = 1$ and $A(\xi) = 0$, where $\xi_i = 0$ for $i > k+1$. Thus,

$$\|\,T - A\,\| \geqslant \|\,T\xi\,\| = \left\| \sum_{n=1}^{n_k} \left(\sum_{i=1}^{n} \beta_i{}^n \xi_i{}^n u_i{}^n\right) + \sum_{n_k+1}^{k+1} \beta_i^{(n_n)} \xi_i^{(n_k)} u_i^{n_k} \right\|$$

$$= \max_{1\leqslant n\leqslant n_k} \left(\sum_{i=1}^{n} |\,\beta_i{}^n \xi_i^{(n)}\,|,\ \sum_{n_k+1}^{k+1} |\,\beta_i^{(n_k)} \xi_i^{(n_k)}\,|\right)$$

$$\geqslant \frac{1}{2} \sum_{i=1}^{k+1} |\,\beta_i \xi_i\,| \geqslant (\beta_{k+1}/2) \sum_{i=1}^{k+1} |\,\xi_i\,| = \beta_{k+1}/2,$$

i.e., $\langle l_1, c_0\rangle$ is a Bernstein pair.

4.13. THEOREM. *If* $1 < p, q < \infty$, *then* $\langle l_p, l_q \rangle$ *is a Bernstein pair.*

Proof. The proof follows from 4.10 since l_q has a shrinking basis for $1 < q < \infty$. Also, the estimates of 2.9 are not good enough to prove that $\langle l_p, l_q \rangle$ is a Bernstein pair for $p < q$. However, a modification of the idea of 3.4 yields an alternate proof. We write $l_p = (\bigoplus l_2(\nu_n))_p$, $l_q = (\bigoplus l_2(\nu_n))_q$, where $\nu_n = n(n+1)/2$. This is possible by [49]. Again, using the technique of [31, Theorem 3.7], we obtain that $\langle l_p, l_q \rangle$ is a Bernstein pair.

Now, we must consider $\mathscr{L}_\infty$-domains and $\mathscr{L}_\infty$-ranges. We first show that the problems reduce to the separable case.

4.14. LEMMA. *Suppose that* E_0 *is embedded in* E *such that* E_0'' *is complemented in* E''. *Then, there is a* $\lambda \geqslant 1$ *such that for any* F *and* $T \in K(F, E_0)$

$$\alpha_n(T) \leqslant \lambda \alpha_n(iT), \qquad \text{for all } n.$$

Here, i denotes the embedding map.

Proof. By 4.7, we have $\alpha_n(T) = \alpha_n(T'') = \alpha_n(Pi''T'') \leqslant \| P \| \alpha_n(i''T'') = \| P \| \alpha_n(iT)$.

Always, $\alpha_n(iT) \leqslant \alpha_n(T)$ and so under the hypotheses of 4.14 we have that $(\alpha_n(T))$ and $(\alpha_n(iT))$ are equivalent.

4.15. COROLLARY. *The lemma is satisfied for* $E_0 \subset E$ *both* $\mathscr{L}_\infty$*-space.*

The corollary follows since the bidual of an $\mathscr{L}_\infty$-space is injective [23]. Now, every $\mathscr{L}_\infty$-space contains a separable $\mathscr{L}_\infty$ subspace and this yields our result for $\mathscr{L}_\infty$-ranges, i.e., if $\langle F, E \rangle$ is a Bernstein pair for all separable $\mathscr{L}_\infty$-spaces, then $\langle F, E \rangle$ is a Bernstein pair for *all* $\mathscr{L}_\infty$-spaces.

Next we show that we need only consider separable $\mathscr{L}_\infty$-domains.

4.15. LEMMA. *Suppose that* E_0 *is a seprable* $\mathscr{L}_\infty$*-subspace of an* $\mathscr{L}_\infty$*-space* E. *If* $\langle E_0, F \rangle$ *is a Bernstein pair, so is* $\langle E, F \rangle$.

Proof. Let $T: E_0 \to F$ such that $(\alpha_n(T))$ is equivalent to (β_n). Since E_0'' is injective, there is a projection $P: E'' \to E_0''$. Since T is compact, T'' maps E'' into JF, the canonical image of F in F''. Let Q denote the restriction of P to E and let $S = J^{-1}T''Q$. Since S extends T, $\alpha_n(S) \geqslant \alpha_n(T)$. On the other hand, $\alpha_n(S) \leqslant \| J^{-1} \| \| Q \| \alpha_n(T'') = \| Q \| \alpha_n(T)$, the last inequality by 4.7, i.e., $(\alpha_n(S))$ is equivalent to $(\alpha_n(T))$.

Now, we wish to reduce the problem to c_0. We recall the following fact, which is immediate from Stegall's local selection theorem [52] (see also [24]).

4.16. LEMMA. *If* E *is a separable* $\mathscr{L}_\infty$*-space, then there is a quotient map* $q: E \to c_0$ *and a projection* $P: E' \to l^1$ *such that* Pq' *is the identity on* l^1.

4.17. THEOREM. *Let E be a separable $\mathscr{L}_\infty$-space. If $\langle c_0, F\rangle$ is a Bernstein pair, then $\langle E, F\rangle$ is a Bernstein pair.*

Proof. Let $T: c_0 \to F$ be such that $(\alpha_n(T))$ is equivalent to (β_n). Let $\hat{T}: E \to F$ be defined by $\hat{T} = Tq$ where q is the special quotient mapping of 4.16. Then, by 4.7, $\alpha_n(T) = \alpha_n(T') = \alpha_n(Pq'T') \leqslant \| P \| \alpha_n(q'T') = \| P \| \alpha_n(Tq) \leqslant \| P \| \| q \| \alpha_n(T)$, i.e.. $((\alpha_n(\hat{T}))$ is equivalent to $((\alpha_n(T)))$.

To handle the case of separable $\mathscr{L}_\infty$-ranges we must recall three facts. The first is a remark of Pelczynski, the second and third are deep results of Stegall [52] and Stegall and Hagler [51].

4.18. *Remark.* 1. If Y is a subspace of X and Y is isomorphic to Z, then there is an $\tilde{X}$ isomorphic to X and containing Z isometrically.

2. The following statements are equivalent:

(a) There is a Φ from E onto F such that $\Phi'(F')$ is complemented in E'.

(b) For any Banach space X, $I \otimes \Phi: \mathscr{F}(X, E) \to \mathscr{F}(X, F)$ is onto.

3. If E is a separable $\mathscr{L}_\infty$-space, then either E' is isomorphic to l^1 or l_1 is isomorphic to a subspace of E (actually $(\oplus l_\infty{}^n)_1$ is a subspace of E).

We can now prove the desired result.

4.19. THEOREM. *For $1 \leqslant p < \infty$, $\langle l_p, E\rangle$ is a Bernstein pair for any separable $\mathscr{L}_\infty$-space E.*

Proof. We distinguish two cases.

1. E' is isomorphic to l^1: In this case E has a shrinking basis [39]. Thus, by 4.10, $\langle l_p, E\rangle$ is a Bernstein pair for $1 < p < \infty$.

To prove that $\langle l_1, E\rangle$ is a Bernstein pair in this case, we proceed as follows. Let $\Phi: E \to c_0$ be the special metric surjection (which exists by the remarks above) and let $T: l_1 \to$ be an operator obtained from Theorem 4.12. Since l_1 has the lifting property, there is an operator $\hat{T}: l_1 \to E$ such that $\Phi\hat{T} = T$. Thus, $\alpha_n(T) \leqslant \| \Phi \| \alpha_n(\hat{T}) = \alpha_n(\hat{T})$. Also by 1.2, $\alpha_n(\hat{T}) = d_n(\hat{T}) \geqslant d_n(\Phi\hat{T}) = d_n(T) = \alpha_n(T)$ and 4.19 follows for E' isomorphic to l^1.

2. l_1 is isomorphic to a subspace of E.

In this case, we can suppose, by the remarks above, that l_1 is isometrically isomorphic to a subspace of E. By Theorem 4.9, $\langle l_p, l_1\rangle$ is a Bernstein pair.

Let $T: l_p \to l_1$ be an operator with $(\alpha_n(T))$ equivalent to (β_n). Let I be the isometry of $l_1 \to E$. We claim that IT has approximation numbers equivalent to (β_n). First, observe that $\alpha_n(IT) \leqslant \alpha_n(T) = \alpha_n(T')$. But $T': l_\infty \to l_{p'}$ is a diagonal map and so by 1.8, $\alpha_n(T') = d_n(T')$. Thus, we have

$$\alpha_n(IT) \leqslant \alpha_n(T') = d_n(T') = d_n(T'I') \leqslant \alpha_n(T'I') = \alpha_n(IT).$$

Thus, $(\alpha_n(IT))$ is equivalent to $(\alpha_n(T))$. (The equality $d_n(T') = d_n(T'I')$ follows since I' is a metric surjection.

We have only the extreme case $\langle E_1, E_2 \rangle$ with both E_1, E_2 $\mathscr{L}_\infty$-spaces, to consider. By 4.17 and the remark after 4.15, we can suppose that $E_1 = c_0$ and E_2 is separable.

4.20. THEOREM. *If E is a separable $\mathscr{L}_\infty$-space, then $\langle c_0, E \rangle$ is a Bernstein pair.*

Proof. Again, we distinguish two cases:

1. E' is isomorphic to l^1. Since E' is separable, E has a shrinking basis [39] and the result follows from 4.10.

2. l^1 is a subspace E. Let $T: c_0 \to l_1$ be an operator corresponding to (β_n). The argument now proceeds as in case 2 of 4.19.

Combining the above results we finally obtain the main result.

4.20. THEOREM. *Let E be an $\mathscr{L}_p$-space and F an $\mathscr{L}_q$-space $1 \leqslant p, q \leqslant +\infty$. Then, $\langle E, F \rangle$ is a Bernstein pair.*

We prove a final result that gives a sufficient condition for two Banach spaces E and F to form a Bernstein pair. The hypotheses may be satisfied by arbitrary Banach spaces, but we have been unable to prove this.

We first introduce some notation.

Let $\varphi = (\varphi_n)$ be a linearly independent sequence in a Banach space E and let $\beta = (\beta_n)$ be a positive, monotonically decreasing null sequence. Then, for each n let

$$\Sigma(\varphi, \beta) = \{x \in E: d(x, [\varphi_1, \ldots, \varphi_n]) \leqslant \beta_n\}.$$

4.21. *Remark.* Let E and F be Banach spaces and $\beta = (\beta_n)$ a null sequence as above. Suppose that there is a linearly independent sequence $\psi = (\psi_n)$ in F and a linearly independent sequence of rank one operators $\varphi = (\varphi_n)$ in $\mathscr{L}(E, F)$ and a $T \in \Sigma(\varphi, \beta)$ such that

$$T(U_E) \supset \Sigma(\psi, \beta).$$

Then, $\langle E, F \rangle$ is a Bernstein pair.

Proof. First, observe that for any linearly independent sequence $x = (x_i)$,

$$d_n(\Sigma(x, \beta)) = \beta_n.$$

Indeed, $d_n(\Sigma(x, \beta)) \leqslant d(x, [x_1, ..., x_n]) \leqslant \beta_n$ for any $x \in \Sigma(x, \beta)$. Also, if $A_n = \beta_n(U_E \cap [x_1, ..., x_{n+1}])$ and $x \in A_n$, then for $m = 1, 2, ..., n$,

$$d(x, [x_1, ..., x_m]) \leqslant \| x \| \leqslant \beta_n \leqslant \beta_m, \quad \text{and} \quad d(x, [x_1, ..., x_m]) = 0, \quad \text{if } m > n, \quad \text{i.e., } A_n \subset \Sigma(x, \beta).$$

By the Krein–Milman–Krasnoselskii theorem, we have $\beta_n = d_n(A_n)$. Thus, $\beta_n = d_n(A_n) \leqslant d_n(\Sigma(x, \beta)) \leqslant \beta_n$ for each n.

By hypothesis, we thus have

$$d_n(T) \geqslant \beta_n .$$

On the other hand, $T \in \Sigma(\varphi, \beta)$ and since $\lim_n \beta_n = 0$, $T \in [\varphi_i]$. Since the φ_i are all rank one operators, we thus have

$$\alpha_n(T) \leqslant d(T, [\varphi_1, ..., \varphi_n]) \leqslant \beta_n .$$

Thus, $\beta_n \leqslant d_n(T) \leqslant \alpha_n(T) \leqslant \beta_n$ and $\langle E, F \rangle$ is a Bernstein pair.

We remark that the fact that $d_n(\Sigma(\varphi, \beta)) = \beta_n$ is well known. Indeed, the sets $\Sigma(\varphi, \beta)$ have been called "full approximation sets" by Kolmogoroff and have been studied in some detail (see, e.g., [25]).

5. Concluding Remarks

Since preparing this paper, we have received a preprint from Pietsch [47]. In his paper, Pietsch develops an axiomatic theory of s-numbers. A few remarks concerning his paper are in order. We recall that Gohberg and Krein [8] define the s-numbers of a bounded linear operator on Hilbert space as the eigenvalues of $(TT^*)^{1/2}$. The approximation numbers and Kolmogoroff diameters generalize this concept to operators between arbitrary Banach spaces. Thus, Pietsch was led to develop a very general theory of s-numbers. More specifically, let $\mathscr{L}$ denote the ideal of all bounded linear operators between Banach spaces and Λ the class of all non-negative sequences. A map s: $\mathscr{L} \to \Lambda$ such that

(1) $\| T \| = S_0(T) \geqslant S_1(T) \geqslant \cdots$,

(2) $S_n(S + T) \leqslant S_n(S) + \| T \|$,

(3) $S_n(RST) \leqslant \| R \| S_n(S) \| T \|$,

(4) rank $T \leqslant n$ implies $S_n(T) = 0$, and

(5) dim $E > n$ implies $S_n(I_E) = 1$,

is called a sequence of s-numbers for the operator T.

This definition is a true generalization of the s-numbers of operators on Hilbert spaces as the following result of Pietsch shows: *If T is a compact operator on a Hilbert space and S any sequence of s-numbers for T then $S_n(T)$ is the $(n+1)$-eigenvalue* of $(TT^*)^{1/2}$. It is interesting that the sequence of approximation numbers $(\alpha_n(T))$ form the largest (under coordinatewise ordering) s-numbers. The fact was also observed by Pietsch.

We conclude by mentioning the overlaps of [47] with our paper. Pietsch has observed that our Theorems 1.1 and 1.2 and the result of Johnson (Theorem 2.7) are valid. He also mentions that Solomjak and Tichomirov [48] have obtained a version of our Theorem 2.12. We have been unable to obtain paper [48] for comparison.

From the remarks in [47] it appears that our proof is much easier than that of [48].

Also, Pietsch has observed that even for operators $T: E \to F$ with $\dim E < +\infty$, it is possible that $d_n(T) \neq d_n(T')$. Thus, there is no analog of 4.7 for Kolmogoroff diameters. In particular, if one considers the notion of a Bernstein pair for Kolmogoroff diameters instead of approximation numbers, there is, at present, no way to obtain results similar to those of Chapter 4.

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